

Hydrodynamic Equations in Quantum Hall Systems at Large Currents*

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Hydrodynamic equations (HDEQs) are derived which describe spatio-temporal evolutions of the electron temperature and the chemical potential of two-dimensional systems in strong magnetic fields in states with large diagonal resistivity appearing at the breakdown of the quantum Hall effect. The derivation is based on microscopic electronic processes consisting of drift motions in a slowly-fluctuating potential and scattering processes due to electron-electron and electron-phonon interactions. In contrast with the usual HDEQs, one of the derived HDEQs has a term with an energy flux perpendicular to the electric field due to the drift motions in the magnetic field. As an illustration, the current distribution is calculated using the derived HDEQs.

KEYWORDS: integer quantum Hall effect, nonlinear transport, theory

§1. Introduction

The diagonal resistivity ρ_{xx} is extremely small in the quantum Hall effect^{1,2)} (QHE) which occurs when the current density j is small. When j is increased up to a critical value, on the other hand, ρ_{xx} increases by several orders of magnitude within a narrow range of j , and the QHE breaks down^{3,4,5)} (reviews are given in refs. 6, 7, 8). Accompanying the breakdown of the QHE, spatial^{9,10,11,12,13)} and temporal^{4,14,15)} evolutions of ρ_{xx} have been observed, and other various phenomena are expected such as pattern formations and oscillatory instabilities which have been extensively studied in bulk semiconductors.¹⁶⁾ Unique phenomena are also expected in the present systems with strong magnetic fields, which have been partly revealed in the experiments.^{9,10,11,12,13)}

Theoretically, on the other hand, a hot-electron model^{3,17,18)} has been proposed as a mechanism of the breakdown of the QHE. It is proposed in this model that the electron heating is responsible for the increase of ρ_{xx} at the breakdown, that is, the electron temperature T_e is the key variable in determining ρ_{xx} . Later the hot-electron model has been supported by observed spatial variations of ρ_{xx} at a constant j ,^{9,10,11,12)} which are understood if we assume that spatial variations of T_e are responsible for those of ρ_{xx} . The importance of T_e has already been suggested by the previous experiments showing a bistability of a small- ρ_{xx} state and a large- ρ_{xx} state at the breakdown,^{3,4,5)} since the presence of such bistability requires another variable, in addition to j , to determine ρ_{xx} . Based on the hot-electron model, a hydrodynamic equation (HDEQ) to calculate spatio-temporal variations of T_e has been proposed for one-dimensional variations in the previous work.¹⁷⁾ However, its validity in the present quantum Hall systems is not known, because it was not derived on the basis of microscopic

electronic processes. The theoretical framework to predict a variety of phenomena in the present systems has not been established.

In this study, we derive a set of HDEQs to describe spatio-temporal variations of the electron temperature T_e and the chemical potential μ in large- ρ_{xx} states on the basis of microscopic processes in a two-dimensional electron system (2DES) in strong magnetic fields in the presence of a slowly-fluctuating potential due to ionized donors. As an example of its applications, we calculate the current distribution. One of the HDEQs in this paper is a generalization of the HDEQ given in our previous paper,¹⁹⁾ which is the equation for T_e in the case of one-dimensional spatial modulations restricted to the current direction.

The organization of this paper is as follows. In §2, three major assumptions employed in deriving the HDEQs are introduced with several minor assumptions, and a model for electronic states and processes, which has been proposed in our previous paper,²⁰⁾ is described. In §3, two HDEQs are derived which describe evolutions of the electron density and the energy density, respectively. In §4, formulas of the electron and energy flux densities as well as the energy loss, appearing in the HDEQs, are derived. In §5, values of coefficients in the HDEQs are estimated, and scaled HDEQs are introduced. The critical electric field is calculated using the estimated values of coefficients in steady and uniform states at even-integer filling factors, and is compared with observed values. In §6, the HDEQs are applied to obtain the current distribution at even-integer filling factors. In §7, conclusions and discussions are given.

§2. Assumptions and Model

2.1 Single-Electron Approximation and Scattering Processes

The potential V_{donor} produced by ionized donors fluctuates in the plane of the 2DES. Its typical length scale,²¹⁾ or the average distance between a potential hill and a neighboring valley, ℓ_{vh} , is $\ell_{\text{vh}} \sim 0.1 \mu\text{m} \gg \ell$ with

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ℓ the magnetic length. V_{donor} is partly screened by the electron-electron interaction V_{ee} . The screened potential calculated in the Hartree approximation^{22, 23)} has an amplitude approximately equal to the Landau level separation, $\hbar\omega_c$. In such a potential, electrons (holes) populate in the $N+1$ th (N th) Landau level, even in a filling factor of $2N$ ($N = 1, 2, \dots$).

In deriving the HDEQs, we take into account processes which are not included in the Hartree approximation. We divide the electron-electron interaction into two terms: $V_{\text{ee}} = \langle V_{\text{ee}} \rangle + (V_{\text{ee}} - \langle V_{\text{ee}} \rangle)$. The first term is the electron-electron interaction averaged with respect to the distribution of other electrons in the Hartree approximation. $\langle V_{\text{ee}} \rangle$ and V_{donor} give the screened potential as calculated by the previous authors.^{22, 23)} We introduce single-electron states in such a mean-field potential. The second term, on the other hand, gives scattering processes between the single-electron states. We consider such scattering processes only in the lowest order of $V_{\text{ee}} - \langle V_{\text{ee}} \rangle$.

2.1.1 Model of electronic states and processes

Electronic states in the slowly-varying screened potential consist of localized states with size $\sim \ell_{\text{vh}}$ around a potential hill or valley and extended states with energy at the center of the broadened Landau level. Each state has a level broadening due to the electron-electron scatterings. An estimate of the level broadening is $\Gamma \sim 0.2\hbar\omega_c$ and that of the inelastic scattering length is $\ell_{\text{ee}} \sim \ell$, according to the estimation in §4.7 at $T_e \sim T_{c1}$ (the definition of T_{c1} is given in §2.1.2). Due to such electron-electron scatterings quantized energy levels of the localized states are smeared out. We therefore label the localized states by a continuous momentum a which is related to a continuous energy in terms of the local electric field. The same electron-electron scatterings destroy the phase coherence of extended states. In such situation it is convenient to introduce wave packets with size $\sim \ell_{\text{ee}}$ by superposing extended states. Here we divide the 2DES into regions so that each region contains one valley or one hill separated by equipotential lines of the fluctuating potential at the energy of extended states. We include each wave packet traveling along the boundary equipotential line in either the neighboring hill or valley region. All states are then labeled by α , i , and a where $\alpha = N\sigma$ labels a Landau level with $N = 1, 2, \dots$ the Landau index and σ the spin, i labels a region, and a labels a momentum.

Processes which transfer electrons between states consist of drift motions and scattering processes. Drift motions perpendicular to the local electric field transfer electrons in extended states between neighboring regions (which are called d-processes), giving the Hall current perpendicular to the macroscopic electric field. Scattering processes we consider are due to electron-electron and electron-phonon interactions. In our model of scattering processes, a transition of an electron is restricted to be within the same level (diagonal in α) and no inter-Landau-level transitions are considered, since the transition rates are much smaller than those of intra-Landau-level transitions.²⁰⁾ In electron-phonon scatter-

ings (which are called p-processes) we consider scatterings within the same region only (diagonal in i). In electron-electron scatterings, we consider the following two types. In the first type, one electron hops between neighboring regions, that is, between a hill and a valley, and another makes a transition within a region (h-processes). In the second type, both electrons make a transition within a region (s-processes). The temporal evolution of the occupation probability $f_{\alpha ia}(t)$ of state αia is given by the Boltzmann-type equation:

$$\frac{\partial f_{\alpha ia}}{\partial t} = \left(\frac{\partial f_{\alpha ia}}{\partial t} \right)_d + \left(\frac{\partial f_{\alpha ia}}{\partial t} \right)_p + \left(\frac{\partial f_{\alpha ia}}{\partial t} \right)_h + \left(\frac{\partial f_{\alpha ia}}{\partial t} \right)_s. \quad (2.1)$$

2.1.2 Characteristic scales

The lower critical current density j_{c1} and the lower critical electric field E_{c1} are defined by the values at which the large- ρ_{xx} state becomes unstable. Values of the current density and the electric field are of the order of j_{c1} and E_{c1} , respectively, in this paper. Similarly, values of T_e are around T_{c1} , which is the value at j_{c1} in the large- ρ_{xx} state.

The length scales in the present system are the magnetic length ℓ , the average distance between a valley and a hill ℓ_{vh} , and ℓ_{c1} defined by $eE_{c1}\ell_{c1} = \hbar\omega_c$. ℓ_{c1} is shown in §5.3 to be the length scale of variations of T_e and μ . The typical sizes are $\ell = 0.01\mu\text{m}$, $\ell_{\text{vh}} \sim 0.1\mu\text{m}$, and $\ell_{c1} \sim 2\mu\text{m}$ at $B = 5\text{T}$ and $E_{c1} = 50\text{V/cm}$, and $\ell \ll \ell_{\text{vh}} \ll \ell_{c1}$. The corresponding time scales are ω_c^{-1} , $\tau_{\text{vh}} = \ell_{\text{vh}}/v_{\text{vh}}$, and $\tau_{c1} = \ell_{c1}/v_{c1}$ where $v_{\text{vh}} = eE_{\text{vh}}\ell^2/\hbar$ is the group velocity at the local electric field between a valley and a hill $E_{\text{vh}} = \hbar\omega_c/e\ell_{\text{vh}}$, and $v_{c1} = eE_{c1}\ell^2/\hbar$. The ratios between the time scales are $\omega_c\tau_{\text{vh}} = \ell_{\text{vh}}^2/\ell^2 \sim 10^2$ and $\omega_c\tau_{c1} = \ell_{c1}^2/\ell^2 \sim 10^4$. Finally the energy scale is $\hbar\omega_c$.

2.2 Local Equilibrium

We assume the local equilibrium in the scale of ℓ_{vh} and introduce the electron temperature $T_e(\mathbf{r}_i, t)$ and the chemical potential $\mu(\mathbf{r}_i, t)$ of the i th region at the position $\mathbf{r}_i = (x_i, y_i)$. In the local equilibrium $f_{\alpha ia}(t)$ becomes

$$f_{\alpha ia}(t) = f(\varepsilon_{\alpha ia} - \mu(\mathbf{r}_i, t), T_e(\mathbf{r}_i, t)), \quad (2.2)$$

where $f(\varepsilon, T) = 1/[\exp(\varepsilon/k_B T) + 1]$ and $\varepsilon_{\alpha ia}$ is the energy of state αia . We neglect deviations of $f_{\alpha ia}(t)$ due to the presence of electron and energy fluxes. With use of such averaged occupation probability, temporal evolutions of the system become irreversible.

To be consistent with the local equilibrium in the scale of ℓ_{vh} , it is necessary that the inelastic scattering length ℓ_{ee} of extended states should be shorter than ℓ_{vh} . The estimation in §4.7 gives $\ell_{\text{ee}} \ll \ell_{\text{vh}}$ when $T_e \sim T_{c1}$. Note that ℓ_{ee} becomes much longer than ℓ_{vh} in the very-low- T_e region, in which the local equilibrium is violated.

It is also necessary to have efficient electron and energy transfers among different levels with different N and σ in order to keep the local equilibrium. Therefore we must restrict our discussion to the case where temporal variations of macroscopic variables are slow enough to

keep the local equilibrium.

2.3 Spatial Averaging

The 2DES actually has a random configuration of potential hills and valleys instead of a periodic array. Due to such randomness, electron and energy densities, as well as electron and energy flux densities, fluctuate from place to place. We here take an average of these quantities over the area larger than the scale of fluctuating potential, ℓ_{vh} . Owing to this averaging, the equations become translationally invariant. To justify this averaging, it is necessary that ℓ_{vh} is much shorter than the length scale ℓ_{var} of variations of T_e and μ . This is approximately satisfied, since it is shown in §5.3 that $\ell_{\text{var}} \sim \ell_{c1}$. Using $\ell_{\text{vh}} \ll \ell_{\text{var}}$, the difference of a quantity between neighboring regions is replaced with the differential in §4.

The averages of the quantities depend in general on the distribution function of the distance between a hill and a valley, which is mainly determined by the average ℓ_{vh} and the standard deviation. We neglect effects of the standard deviation in the calculation in the present paper.

2.4 Other Assumptions

We also make minor assumptions in order to simplify the HDEQs, although it may not be difficult to go beyond them.

(a) We neglect the change of electronic states (in the single-electron approximation) with the applied electric field since $E_{c1} \sim 0.1E_{\text{vh}}$.

(b) The screened potential depends on T_e because the local electron density depends on T_e . We neglect such dependence by considering only the temperature range around T_{c1} .

(c) We neglect the broadening of energy spectra of extended states due to electron-electron scattering processes and the resulting decrease of the activation energy, which are of higher order in electron-electron interactions.

(d) In addition to the valley-hill hopping processes which we have introduced above, elastic tunneling processes giving valley-valley and hill-hill transitions of an electron also contribute to the current along the electric field and consequently to ρ_{xx} . We neglect such tunneling processes in this paper, for simplicity.

(e) In calculating terms in the HDEQs, we assume that the change in energy of an electron at a scattering process $\Delta\varepsilon \ll \pi k_B T_e$ ($\Delta\varepsilon \sim eE_{\text{vh}}\ell$ as shown in our previous paper²⁰⁾) and that the lattice temperature $T_L = 0$, and neglect the energy dependence of the density of states and the spin splitting.

§3. Hydrodynamic Equations

3.1 Macroscopic Variables

Macroscopic variables in the present system are the electron temperature $T_e(\mathbf{r}, t)$, the chemical potential $\mu(\mathbf{r}, t)$, and the electric field $\mathbf{E}(\mathbf{r}, t)$, with $\mathbf{r} = (x, y)$. We introduce a reference energy $\varepsilon_m(\mathbf{r}, t)$ defined by $\varepsilon_m = (\varepsilon_{N\uparrow} + \varepsilon_{N+1\downarrow})/2$, where ε_α is the energy of extended states in the α th level, and the indices N and

$N+1$ are those of the two Landau levels in the both sides of μ . The deviation of μ from ε_m is introduced and is denoted by $\mu_r = \mu - \varepsilon_m$. The electric field is given by $\mathbf{E} = (E_x, E_y) = \nabla \varepsilon_m / e$ where $\nabla = (\partial/\partial x, \partial/\partial y)$ and $e > 0$.

T_e and μ_r are intensive variables corresponding to the energy density U and the electron density n , respectively, and are determined by the evolution equation of U and that of n , which are derived in the next two subsections. The increase of T_e produces the increase of ρ_{xx} , and the change of μ_r at high T_e gives the deviation of ρ_{xy} from the quantized value, as shown in eq.(4.17).

3.2 Equation of the Charge Conservation

The electron density averaged over regions within area S satisfying $\ell_{\text{vh}} \ll S^{1/2} \ll \ell_{c1}$ is

$$n(\mathbf{r}, t) = \frac{1}{S} \sum_{\alpha, i, a} f_{\alpha ia}. \quad (3.1)$$

Here the integration over a continuous momentum a is denoted by the summation, for simplicity. The drift and hopping processes give electron transfers between neighboring regions. Since the change of n is given only by electron transfers between the area S and the outside through the boundary L , we have

$$S \frac{\partial n}{\partial t} = \sum_{\alpha, i \in S, a} \frac{\partial f_{\alpha ia}}{\partial t} = - \sum_{i \in L} J_i, \quad (3.2)$$

with J_i the number of electrons going from the i th region to the outside of S per unit time. Introducing the averaged electron flux density \mathbf{j}_n , we have

$$\sum_{i \in L} J_i = \int_L \mathbf{j}_n \cdot \mathbf{n} dL = \int_S \nabla \cdot \mathbf{j}_n dS, \quad (3.3)$$

where \mathbf{n} is the unit vector perpendicular to the boundary L directed to the outside, and the two-dimensional Gauss theorem is used in the second equality. Since the variation of $\nabla \cdot \mathbf{j}_n$ within S is negligible, we obtain the equation of the charge conservation:

$$\frac{\partial n}{\partial t} = - \nabla \cdot \mathbf{j}_n. \quad (3.4)$$

The electron flux density \mathbf{j}_n is given by

$$\mathbf{j}_n = \mathbf{j}_n^d + \mathbf{j}_n^h, \quad (3.5)$$

where \mathbf{j}_n^d and \mathbf{j}_n^h are the electron flux densities due to drift and hopping processes, respectively.

3.3 Equation of the Energy Conservation

The energy density due to the thermal activation, averaged over regions within area S satisfying $\ell_{\text{vh}} \ll S^{1/2} \ll \ell_{c1}$, is

$$U(\mathbf{r}, t) = \frac{1}{S} \sum_{\alpha, i, a} (\varepsilon_{\alpha ia} - \mu_i)(f_{\alpha ia} - f_{\alpha ia}^0), \quad (3.6)$$

where $\mu_i = \mu(\mathbf{r}_i, t)$ and $f_{\alpha ia}^0$ is the occupation probability in the local equilibrium at $T_e = 0$: $f_{\alpha ia}^0 = f(\varepsilon_{\alpha ia} - \mu_i, 0)$. Terms with $\varepsilon_{\alpha ia} > \mu_i$ express electron excitations, and

those with $\varepsilon_{\alpha ia} < \mu_i$ express hole excitations:

$$(\varepsilon_{\alpha ia} - \mu_i)(f_{\alpha ia} - f_{\alpha ia}^0) = \begin{cases} (\varepsilon_{\alpha ia} - \mu_i)f_{\alpha ia}, & \varepsilon_{\alpha ia} > \mu_i \\ (\mu_i - \varepsilon_{\alpha ia})(1 - f_{\alpha ia}), & \varepsilon_{\alpha ia} < \mu_i \end{cases} . \quad (3.7)$$

U is considered as a function of the electron distribution described by μ_r and $f_{\alpha ia}$. In $\partial U/\partial t$, terms with $\partial \mu_r/\partial t$ are neglected since μ_r changes much slower in time than $f_{\alpha ia}$. Then $\partial U/\partial t$ is given, using the Boltzmann-type equation eq.(2.1), as

$$\frac{\partial U}{\partial t} = \left(\frac{\partial U}{\partial t} \right)_d + \left(\frac{\partial U}{\partial t} \right)_p + \left(\frac{\partial U}{\partial t} \right)_{h1} + \left(\frac{\partial U}{\partial t} \right)_{h2} + \left(\frac{\partial U}{\partial t} \right)_s . \quad (3.8)$$

Here we have decomposed the term due to h-processes into two terms: h1 and h2 denote the contribution from hopping electrons and that from electrons scattered within a region, respectively.

The drift and hopping processes give energy transfers between regions. The transferred energy $|\varepsilon_{\alpha ia} - \mu_i|$ is $\varepsilon_{ra} - \mu_r$ for electron excitations and $\mu_r - \varepsilon_{ra}$ for hole excitations with $\varepsilon_{ra} = \varepsilon_\alpha - \varepsilon_m$. Although $\varepsilon_{\alpha ia} - \varepsilon_m$ of the initial and final states in the hopping processes has fluctuations around ε_{ra} due to the fluctuating potential, such fluctuations are canceled by those in the energy of the scattered electron in the calculation of the h2 term (see below). The s-processes also give energy transfers between regions. The change of U due to such energy transfers is given by

$$\left(\frac{\partial U}{\partial t} \right)_d + \left(\frac{\partial U}{\partial t} \right)_{h1} + \left(\frac{\partial U}{\partial t} \right)_s = -\nabla \cdot \mathbf{j}_U , \quad (3.9)$$

following the procedure in the previous subsection. Here the averaged energy flux density \mathbf{j}_U is given by

$$\mathbf{j}_U = \mathbf{j}_U^d + \mathbf{j}_U^h + \mathbf{j}_U^s , \quad (3.10)$$

where \mathbf{j}_U^d , \mathbf{j}_U^h , and \mathbf{j}_U^s are components due to drift, hopping, and scattering (without hopping) processes, respectively.

The 2DES gains an energy from the electric field through the energy change $\Delta\varepsilon$ of the scattered electron in the h-processes. Neglecting the fluctuations in $\Delta\varepsilon$ (see above), $\Delta\varepsilon = (-e)\mathbf{E} \cdot \Delta\mathbf{r}$ with $\Delta\mathbf{r}$ the change in position of the hopping electron. By summing over all hopping processes within S , we obtain

$$\left(\frac{\partial U}{\partial t} \right)_{h2} = (-e)\mathbf{E} \cdot \mathbf{j}_n^h . \quad (3.11)$$

Electron-phonon scatterings give changes of energy within each region and result in the energy loss after the statistical averaging when the lattice temperature is lower than T_e :

$$\left(\frac{\partial U}{\partial t} \right)_p = -P_L , \quad (3.12)$$

where $P_L > 0$ is the energy loss per unit area per unit time.

Then we obtain

$$\frac{\partial U}{\partial t} = -\nabla \cdot \mathbf{j}_U + (-e)\mathbf{E} \cdot \mathbf{j}_n - P_L , \quad (3.13)$$

where we have used $\mathbf{E} \cdot \mathbf{j}_n^d = 0$.

3.4 Boundary Conditions

We impose boundary conditions at the edge of the 2DES in solving the HDEQs. In the case of the boundary between the 2DES and the depleted region, they are

$$\mathbf{j}_n \cdot \mathbf{n}_b = 0 , \quad \mathbf{j}_U \cdot \mathbf{n}_b = 0 , \quad (3.14)$$

where \mathbf{n}_b is the unit vector perpendicular to the boundary. Another boundary condition is that the total current is given by the external circuit.

§4. Evaluation of Terms in the HDEQs

4.1 Dimensionless HDEQs

Here we introduce dimensionless units in which the length and the time are scaled by ℓ and ω_c^{-1} , respectively. Dimensionless variables are defined as $\tilde{\mathbf{r}} = \mathbf{r}/\ell$, $\tilde{\nabla} = \ell\nabla$, $\tilde{t} = \omega_c t$, and $\tilde{\mathbf{E}} = e\ell\mathbf{E}/\hbar\omega_c$. We use $\hbar\omega_c$ to scale energies and introduce $\tilde{T}_e = k_B T_e/\hbar\omega_c$, $\tilde{\mu}_r = \mu_r/\hbar\omega_c$, and $\tilde{\varepsilon}_{\alpha ia} = \varepsilon_{\alpha ia}/\hbar\omega_c$.

In the present units the equation of the charge conservation becomes

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} = -\tilde{\nabla} \cdot \tilde{\mathbf{j}}_n , \quad (4.1)$$

with $\tilde{n} = \ell^2 n$ and $\tilde{\mathbf{j}}_n = (\ell/\omega_c)\mathbf{j}_n$. The equation of the energy conservation becomes

$$\frac{\partial \tilde{U}}{\partial \tilde{t}} = -\tilde{\nabla} \cdot \tilde{\mathbf{j}}_U - \tilde{\mathbf{E}} \cdot \tilde{\mathbf{j}}_n - \tilde{P}_L , \quad (4.2)$$

with $\tilde{U} = (\ell^2/\hbar\omega_c)U$, $\tilde{\mathbf{j}}_U = (\ell/\hbar\omega_c^2)\mathbf{j}_U$, and $\tilde{P}_L = (\ell^2/\hbar\omega_c^2)P_L$.

4.2 Time Derivatives of Electron and Energy Densities

The electron density is given in the local equilibrium by

$$n(\mathbf{r}, t) = 2 \int_{\varepsilon_b}^{\infty} d\varepsilon \rho(\varepsilon) f(\varepsilon - \mu, T_e) , \quad (4.3)$$

where $\rho(\varepsilon)$ is the density of states per spin per unit area and ε_b is the energy of the lowest state. Using the assumption that $\rho(\varepsilon)$ is constant: $\rho = 1/(2\pi\ell^2\hbar\omega_c)$, we obtain $n(\mathbf{r}, t) = 2\rho(\mu - \varepsilon_b) = 2\rho(\mu_r + \varepsilon_m - \varepsilon_b)$ as long as $f(\varepsilon_b - \mu, T_e) \approx 1$. In the dimensionless units, we have

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} = \frac{1}{\pi} \frac{\partial \tilde{\mu}_r}{\partial \tilde{t}} . \quad (4.4)$$

Similarly, the energy density is given in the local equilibrium by

$$U(\mathbf{r}, t) = 2 \int_{-\infty}^{\infty} d\varepsilon \rho(\varepsilon) (\varepsilon - \mu) \times [f(\varepsilon - \mu, T_e) - f(\varepsilon - \mu, 0)] . \quad (4.5)$$

Since ρ is assumed to be constant, $\tilde{U}(\mathbf{r}, t)$ does not de-

pend on $\tilde{\mu}_r$, and we have

$$\frac{\partial \tilde{U}}{\partial \tilde{t}} = \frac{\partial \tilde{U}}{\partial \tilde{T}_e} \frac{\partial \tilde{T}_e}{\partial \tilde{t}} = \tilde{C}_e \frac{\partial \tilde{T}_e}{\partial \tilde{t}}, \quad (4.6)$$

with the dimensionless specific heat $\tilde{C}_e = \frac{\pi}{3} \tilde{T}_e$.

4.3 Energy Loss

The energy loss due to electron-phonon scatterings is given by

$$\tilde{P}_L = \frac{1}{\tilde{S}} \sum_{\alpha, i, a, b} (\tilde{\varepsilon}_{\alpha ia} - \tilde{\varepsilon}_{\alpha ib}) f_{\alpha ia} (1 - f_{\alpha ib}) \tilde{W}_{\alpha ia \rightarrow ib}^p, \quad (4.7)$$

where $\tilde{S} = S/\ell^2$ and $\tilde{W}_{\alpha ia \rightarrow ib}^p$ is the number during the time ω_c^{-1} of electron-phonon scatterings, in which an electron makes a transition $\alpha ia \rightarrow \alpha ib$, and a phonon is created or destructed depending on the sign of $\varepsilon_{\alpha ia} - \varepsilon_{\alpha ib}$.

We evaluate \tilde{P}_L when the lattice temperature $T_L = 0$. Here we assume that $|\varepsilon_{\alpha ia} - \varepsilon_{\alpha ib}| \ll \pi k_B T_e$, which is approximately satisfied since $|\varepsilon_{\alpha ia} - \varepsilon_{\alpha ib}| \sim e E_{vh} \ell \sim 0.1 \hbar \omega_c$ and $\pi k_B T_e \sim \hbar \omega_c$ at $T_e = T_{c1}$ from eq.(5.3). Using this assumption, we have $f_{\alpha ia} (1 - f_{\alpha ib}) \approx f_{\alpha ia} (1 - f_{\alpha ia}) = -k_B T_e \partial f(\varepsilon_{\alpha ia} - \mu, 0) / \partial \varepsilon$, and then obtain

$$\tilde{P}_L = C_p \tilde{T}_e, \quad (4.8)$$

with

$$\begin{aligned} C_p &= \frac{1}{\tilde{S}} \sum_{\alpha, i, a} (-1) \frac{\partial f}{\partial \varepsilon} (\varepsilon_{\alpha ia} - \mu, 0) \\ &\times \sum_b (\varepsilon_{\alpha ia} - \varepsilon_{\alpha ib}) \tilde{W}_{\alpha ia \rightarrow ib}^p. \end{aligned} \quad (4.9)$$

We here neglect the weak αia dependence of $\tilde{W}_{\alpha ia \rightarrow ib}^p$ ($\tilde{W}_{\alpha ia \rightarrow ib}^p$ depends more strongly on $\varepsilon_{\alpha ib} - \varepsilon_{\alpha ia}$ through the overlap integral between the states). Using also the assumption of no energy dependence of the density of states, C_p becomes independent of T_e and μ_r , and is given by

$$C_p = \frac{1}{\pi} \sum_b (\tilde{\varepsilon}_{\alpha ia} - \tilde{\varepsilon}_{\alpha ib}) \tilde{W}_{\alpha ia \rightarrow ib}^p. \quad (4.10)$$

4.4 Drift Components of Electron and Energy Fluxes

The electron flux density, averaged in a macroscopic scale, due to drift processes, is given by

$$\tilde{\mathbf{j}}_n^d = \sum_{\alpha} \tilde{\mathbf{j}}_{n\alpha}^d, \quad (4.11)$$

where $\tilde{\mathbf{j}}_{n\alpha}^d$ is the averaged electron flux density carried by electrons in the level α . The energy flux density, on the other hand, is given by

$$\begin{aligned} \tilde{\mathbf{j}}_U^d &= \sum_{\alpha (\varepsilon_{\alpha} > \mu)} (\tilde{\varepsilon}_{r\alpha} - \tilde{\mu}_r) \tilde{\mathbf{j}}_{n\alpha}^d \\ &+ \sum_{\alpha (\varepsilon_{\alpha} < \mu)} (\tilde{\mu}_r - \tilde{\varepsilon}_{r\alpha}) (\tilde{\mathbf{j}}_{n\alpha 1}^d - \tilde{\mathbf{j}}_{n\alpha}^d), \end{aligned} \quad (4.12)$$

where $\tilde{\mathbf{j}}_{n\alpha 1}^d$ is the value of $\tilde{\mathbf{j}}_{n\alpha}^d$ when the level α is completely filled by electrons.

Since $\tilde{\mathbf{j}}_{n\alpha}^d$ is an averaged flux density, it is only from

extended states and is given by

$$\tilde{\mathbf{j}}_{n\alpha}^d = f_{\alpha} \tilde{\mathbf{j}}_{n\alpha 1}^d, \quad (4.13)$$

$$f_{\alpha} = f(\varepsilon_{r\alpha} - \mu_r, T_e). \quad (4.14)$$

It was shown by MacDonald *et al.*²⁴⁾ that the local electron flux density of the filled Landau level due to the drift motion in the local electric field \mathbf{E}_{loc} is independent of the Landau index (therefore denoted as $\tilde{\mathbf{j}}_{n1}^{d, loc}$) when \mathbf{E}_{loc} varies in the scale $\gg \ell$, and that $\tilde{\mathbf{j}}_{n1}^{d, loc} = (e/h) \mathbf{E}_{loc} \times \mathbf{e}_z$, with \mathbf{e}_z the unit vector along z ($\mathbf{B} = B \mathbf{e}_z$ with $B > 0$). In the dimensionless units, $\tilde{\mathbf{j}}_{n1}^d = \frac{1}{2\pi} \tilde{\mathbf{E}}_{loc} \times \mathbf{e}_z$. By averaging this equation over S , we obtain

$$\tilde{\mathbf{j}}_{n1}^d = \frac{1}{2\pi} \tilde{\mathbf{E}} \times \mathbf{e}_z = \frac{1}{2\pi} (\tilde{E}_y, -\tilde{E}_x), \quad (4.15)$$

and $\tilde{\mathbf{j}}_{n\alpha 1}^d = \tilde{\mathbf{j}}_{n1}^d$. The electron flux density $\tilde{\mathbf{j}}_n^d$ is then given by

$$\tilde{\mathbf{j}}_n^d = -\tilde{\sigma}_{xy} (\tilde{E}_y, -\tilde{E}_x), \quad (4.16)$$

$$\tilde{\sigma}_{xy} = \sigma_{xy}/(e^2/\hbar) = -\frac{1}{2\pi} \sum_{\alpha} f_{\alpha}. \quad (4.17)$$

When $\mu_r = 0$, $\tilde{\sigma}_{xy} = -\frac{N}{\pi}$ with N the number of filled Landau levels.

The energy flux density, on the other hand, is given by

$$\tilde{\mathbf{j}}_U^d = \tilde{u}_{xy} (\tilde{E}_y, -\tilde{E}_x), \quad (4.18)$$

$$\begin{aligned} \tilde{u}_{xy} &= \frac{1}{2\pi} \sum_{\alpha (\varepsilon_{\alpha} > \mu)} (\tilde{\varepsilon}_{r\alpha} - \tilde{\mu}_r) f_{\alpha} \\ &+ \frac{1}{2\pi} \sum_{\alpha (\varepsilon_{\alpha} < \mu)} (\tilde{\mu}_r - \tilde{\varepsilon}_{r\alpha}) (1 - f_{\alpha}). \end{aligned} \quad (4.19)$$

When we consider only the two Landau levels near μ , assume $\mu_r = 0$, and use the assumption of no spin splitting, we obtain $\tilde{u}_{xy} = \frac{1}{\pi} f_1$ with

$$f_1 = \left[\exp \left(1/2\tilde{T}_e \right) + 1 \right]^{-1}. \quad (4.20)$$

4.5 Hopping Components of Electron and Energy Fluxes

The electron and energy flux densities, averaged in a macroscopic scale, due to electron-electron scatterings (h-processes) are given by

$$\tilde{\mathbf{j}}_n^h = \frac{1}{\tilde{S}} \sum_{\alpha, \beta, i, k, l, a, b, c, d} (\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k) \tilde{J}_{\beta k c \rightarrow l d}^{\alpha ia \rightarrow ib}, \quad (4.21)$$

$$\tilde{\mathbf{j}}_U^h = \frac{1}{\tilde{S}} \sum_{\alpha, \beta, i, k, l, a, b, c, d} (\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r) (\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k) \tilde{J}_{\beta k c \rightarrow l d}^{\alpha ia \rightarrow ib}, \quad (4.22)$$

where $\tilde{S} = S/\ell^2$ and

$$\tilde{J}_{\beta k c \rightarrow l d}^{\alpha ia \rightarrow ib} = f_{\alpha ia} (1 - f_{\alpha ib}) f_{\beta k c} (1 - f_{\beta l d}) \tilde{W}_{\beta k c \rightarrow l d}^{\alpha ia \rightarrow ib}, \quad (4.23)$$

with $\tilde{W}_{\beta k c \rightarrow l d}^{\alpha ia \rightarrow ib}$ the number of scatterings during the period ω_c^{-1} . Regions k and l are the nearest neighbors. For hole excitations ($\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r < 0$), the above formula of $\tilde{\mathbf{j}}_U^h$ is consistent with the picture that each hole excitation carries the energy of $\tilde{\mu}_r - \tilde{\varepsilon}_{r\beta}$, since the hole flux is opposite in sign to the electron flux.

We rewrite $\tilde{\mathbf{j}}_n^h$ as

$$\tilde{\mathbf{j}}_n^h = \frac{1}{2S} \sum_{\alpha, \beta, i, k, l, a, b, c, d} (\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k) (\tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} - \tilde{J}_{\beta ld \rightarrow kc}^{\alpha ib \rightarrow ia}) , \quad (4.24)$$

and $\tilde{\mathbf{j}}_U^h$ similarly, where changes of labels are made in the term with $\tilde{J}_{\beta ld \rightarrow kc}^{\alpha ib \rightarrow ia}$: $a \leftrightarrow b$ and $kc \leftrightarrow ld$. Since the two terms are forward and backward transitions among the same set of states, $\tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} - \tilde{J}_{\beta ld \rightarrow kc}^{\alpha ib \rightarrow ia}$ is the net number (during ω_c^{-1}) of electrons which hop from c in k to d in l , and is induced by $\nabla\mu$ and ∇T_e . In the first order of $\nabla\mu$ and ∇T_e , $\tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} - \tilde{J}_{\beta ld \rightarrow kc}^{\alpha ib \rightarrow ia}$ is given by

$$\begin{aligned} & \tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} - \tilde{J}_{\beta ld \rightarrow kc}^{\alpha ib \rightarrow ia} \\ &= [-\tilde{T}_e^{-1}(\tilde{\mu}_l - \tilde{\mu}_k) + (\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r)(\tilde{T}_l^{-1} - \tilde{T}_k^{-1})] \tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} , \\ &= (\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k) \cdot [-\tilde{T}_e^{-1}\tilde{\nabla}\tilde{\mu} + (\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r)\tilde{\nabla}\tilde{T}_e^{-1}] \tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} , \end{aligned} \quad (4.25)$$

where $\tilde{\mu}_k = \mu(\mathbf{r}_k, t)/\hbar\omega_c$, $\tilde{T}_k = \tilde{T}_e(\mathbf{r}_k, t)$, and $\tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib}$ in the right should be evaluated in the absence of $\nabla\mu$ and ∇T_e . Higher-order terms are negligible when $|\nabla\mu|\ell_{vh}/k_B T_e \ll 1$ and $|\nabla T_e|\ell_{vh}/2T_e\tilde{T}_e \ll 1$. These conditions are approximately satisfied at $E = E_{c1}$ and $T_e = T_{c1}$: since the estimated value of E_{c1} in eq.(5.6) and that of T_{c1} in eq.(5.3) give $|\nabla\mu|\ell_{vh}/k_B T_e \sim 0.1$ and $|\nabla T_e|\ell_{vh}/2T_e\tilde{T}_e \sim 0.1$. We also have made an approximation that $\varepsilon_{\beta kc} - \mu_k, \varepsilon_{\beta ld} - \mu_l \approx \varepsilon_{r\beta} - \mu_r$.

We then have

$$\begin{aligned} \tilde{\mathbf{j}}_n^h &= \frac{1}{2S} \sum_{\alpha, \beta, k, l} (\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k) \\ &\times (\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k) \cdot [-\tilde{T}_e^{-1}\tilde{\nabla}\tilde{\mu} + (\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r)\tilde{\nabla}\tilde{T}_e^{-1}] \\ &\times \sum_{i, a, b, c, d} \tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} . \end{aligned} \quad (4.26)$$

Here we make the integration over θ_{lk} , the polar coordinate of $\mathbf{r}_l - \mathbf{r}_k = r_{lk} \exp(i\theta_{lk}) = (x_{lk}, y_{lk})$. The dependence on $\mathbf{r}_l - \mathbf{r}_k$ appears through $\tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib}$ and $\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k$, but the θ_{lk} dependence appears only through $\tilde{\mathbf{r}}_l - \tilde{\mathbf{r}}_k$, since $\sum_{i, a, b, c, d} \tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib}$ has no dependence on θ_{lk} . We assume the uniform distribution of θ_{lk} , in which we have $\langle x_{lk}y_{lk} \rangle = 0$ and $\langle (x_{lk})^2 \rangle = \langle (y_{lk})^2 \rangle = (r_{lk})^2/2$, with $\langle \dots \rangle \equiv \frac{1}{2\pi} \int_0^{2\pi} \dots d\theta_{lk}$. We also make an approximation that $r_{lk} \approx \ell_{vh}$.

Then we obtain the formulas of $\tilde{\mathbf{j}}_n^h$ and $\tilde{\mathbf{j}}_U^h$:

$$\tilde{\mathbf{j}}_n^h = L_{n\mu} \left(-\tilde{T}_e^{-1}\tilde{\nabla}\tilde{\mu} \right) + L_{nT} \tilde{\nabla}\tilde{T}_e^{-1} , \quad (4.27)$$

$$\tilde{\mathbf{j}}_U^h = L_{U\mu} \left(-\tilde{T}_e^{-1}\tilde{\nabla}\tilde{\mu} \right) + L_{UT} \tilde{\nabla}\tilde{T}_e^{-1} , \quad (4.28)$$

with

$$L_{n\mu} = L_0 , \quad L_{nT} = L_{U\mu} = L_1 , \quad L_{UT} = L_2 , \quad (4.29)$$

$$L_m = \frac{\ell_{vh}^2}{4S} \sum_{\alpha, \beta, i, k, l, a, b, c, d} (\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r)^m \tilde{J}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} , \quad (m = 0, 1, 2) , \quad (4.30)$$

where $\nabla\mu = e\mathbf{E} + \nabla\mu_r \approx e\mathbf{E}$ and in dimensionless units $\tilde{\nabla}\tilde{\mu} \approx \tilde{\mathbf{E}}$.

Here we assume $|\varepsilon_{\alpha ia} - \varepsilon_{\alpha ib}| \ll \pi k_B T_e$. Then we have $f_{\alpha ia}(1 - f_{\alpha ib}) \approx f_{\alpha ia}(1 - f_{\alpha ia}) = -k_B T_e \partial f(\varepsilon_{\alpha ia} - \mu, T_e)/\partial\varepsilon$ and $f_{\beta kc}, f_{\beta ld} \approx f_\beta$. We neglect weak dependences of $\tilde{W}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib}$ on α, i, a , and β , for simplicity. We then obtain, by performing the summation over α, i, a ,

$$L_m = \frac{1}{4} C_h \tilde{T}_e \sum_\beta (\tilde{\varepsilon}_{r\beta} - \tilde{\mu}_r)^m f_\beta (1 - f_\beta) . \quad (4.31)$$

Here

$$C_h = \frac{\tilde{\ell}_{vh}^2}{\pi} \sum_{k, l, b, c, d} \tilde{W}_{\beta kc \rightarrow ld}^{\alpha ia \rightarrow ib} , \quad (4.32)$$

with $\tilde{\ell}_{vh} \equiv \ell_{vh}/\ell$, has no dependence on T_e and μ_r .

When we consider only the two Landau levels near μ , assume $\mu_r = 0$, and use the assumption of no spin splitting, we obtain $L_{n\mu} = C_h \tilde{T}_e f_1(1 - f_1)$, $L_{nT} = L_{U\mu} = 0$, and $L_{UT} = \frac{1}{4} C_h \tilde{T}_e f_1(1 - f_1)$. In the present case $L_{nT} = 0$ because energy levels of extended states are symmetric with respect to μ and electron fluxes induced by ∇T_e are canceled between the upper and lower Landau levels, and $L_{U\mu} = 0$ because electrons and holes move in opposite directions by hopping processes in the presence of \mathbf{E} , in contrast with drift processes, and energy fluxes due to such hopping processes are canceled between the upper and lower Landau levels. The electron and energy flux densities in this case are given in terms of the electrical conductivity σ_{xx} and the thermal conductivity κ_h as

$$\tilde{\mathbf{j}}_n^h = -\tilde{\sigma}_{xx} \tilde{\mathbf{E}} , \quad (4.33)$$

$$\tilde{\mathbf{j}}_U^h = -\tilde{\kappa}_h \tilde{\nabla}\tilde{T}_e , \quad (4.34)$$

with

$$\tilde{\sigma}_{xx} = \sigma_{xx}/(e^2/\hbar) = C_h f_1(1 - f_1) , \quad (4.35)$$

$$\tilde{\kappa}_h = \kappa_h/(k_B \omega_c) = \frac{1}{4} \tilde{T}_e^{-1} \tilde{\sigma}_{xx} . \quad (4.36)$$

4.6 S-process Component of Energy Flux

The energy flux density, averaged in a macroscopic scale, due to electron-electron scatterings without hoppings, is

$$\tilde{\mathbf{j}}_U^s = \frac{1}{2S} \sum_{\alpha, \beta, i, k, a, b, c, d} (\tilde{\varepsilon}_{\alpha ib} - \tilde{\varepsilon}_{\alpha ia})(\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_k) \tilde{J}_{\beta kc \rightarrow kd}^{\alpha ia \rightarrow ib} , \quad (4.37)$$

where

$$\tilde{J}_{\beta kc \rightarrow kd}^{\alpha ia \rightarrow ib} = f_{\alpha ia}(1 - f_{\alpha ib}) f_{\beta kc}(1 - f_{\beta kd}) \tilde{W}_{\beta kc \rightarrow kd}^{\alpha ia \rightarrow ib} . \quad (4.38)$$

The factor 1/2 in the formula of $\tilde{\mathbf{j}}_U^s$ corrects the double counting: $i = 1, k = 2$ and $i = 2, k = 1$, for example. Following the derivation of $\tilde{\mathbf{j}}_U^h$ in the previous subsection, we obtain

$$\tilde{\mathbf{j}}_U^s = -\tilde{\kappa}_s \tilde{\nabla}\tilde{T}_e , \quad (4.39)$$

when $|\nabla T_e|\ell_{vh}/T_e \ll 1$, with the thermal conductivity

$$\tilde{\kappa}_s = \frac{1}{8S\tilde{T}_e^2} \sum_{\alpha, \beta, i, k, a, b, c, d} \tilde{J}_{\beta kc \rightarrow kd}^{\alpha ia \rightarrow ib} (\tilde{\varepsilon}_{\alpha ib} - \tilde{\varepsilon}_{\alpha ia})^2 (\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_k)^2 . \quad (4.40)$$

Here we assume that $|\varepsilon_{\alpha ia} - \varepsilon_{\alpha ib}| \ll \pi k_B T_e$ and neglect the weak α, β, a, c dependence of $\tilde{W}_{\beta kc \rightarrow kd}^{\alpha ia \rightarrow ib}$. We restrict

the calculation to the case of $\mu_r = 0$ and no spin splitting. Then $\tilde{\kappa}_s$ at $\mu_r = 0$, which is denoted by $\tilde{\kappa}_{s0}$, is given by

$$\tilde{\kappa}_{s0} = \frac{\tilde{\ell}_{vh}^4}{8\pi^2 \tilde{S}} \sum_{i,k,b,d} \tilde{W}_{\beta k c \rightarrow k d}^{\alpha i a \rightarrow i b} (\tilde{\varepsilon}_{\alpha i b} - \tilde{\varepsilon}_{\alpha i a})^2 (\tilde{\mathbf{r}}_i - \tilde{\mathbf{r}}_k)^2 . \quad (4.41)$$

$\tilde{\kappa}_{s0}$ has no dependence on T_e .

4.7 Inelastic scattering time

Here we calculate the inelastic scattering time $\tau_{\alpha i a}$ of state $\alpha i a$ due to electron-electron scatterings. Considering s-processes, $\tau_{\alpha i a}$ is given by

$$\frac{1}{\omega_c \tau_{\alpha i a}} = \sum_{\beta, k, b, c, d} (1 - f_{\alpha i b}) f_{\beta k c} (1 - f_{\beta k d}) \tilde{W}_{\beta k c \rightarrow k d}^{\alpha i a \rightarrow i b} . \quad (4.42)$$

Following the evaluation of $\tilde{\kappa}_s$, we assume $|\varepsilon_{\alpha i a} - \varepsilon_{\alpha i b}| \ll \pi k_B T_e$ and neglect the β, c dependence of $\tilde{W}_{\beta k c \rightarrow k d}^{\alpha i a \rightarrow i b}$. Then we obtain, when $\mu_r = 0$,

$$\frac{1}{\omega_c \tau_{\alpha i a}} = \frac{1}{\pi} \tilde{\ell}_{vh}^2 \tilde{T}_e (1 - f_{\alpha i a}) \sum_{k, b, d} \tilde{W}_{\beta k c \rightarrow k d}^{\alpha i a \rightarrow i b} . \quad (4.43)$$

An estimation gives $1/\omega_c \tau_{\alpha i a} \sim 0.2$ for an energy $\varepsilon_{\alpha i a} - \varepsilon_m \sim \hbar \omega_c / 2$ at $\mu_r = 0$, $B = 5T$, and $T_e = T_{c1}$ (the value of T_{c1} is given in eq.(5.3)). Then the level broadening Γ is estimated to be $\Gamma = \hbar / \tau_{\alpha i a} \sim 0.2 \hbar \omega_c$. The inelastic scattering length defined by $\ell_{ee} = \tau_{\alpha i a} v_{vh}$ is estimated to be $\ell_{ee}/\ell = (\omega_c \tau_{\alpha i a}) e E_{vh} \ell / \hbar \omega_c \sim 1$.

§5. Estimation of Coefficients and Scaled HD-EQs

5.1 Uniform Steady States at Even-Integer Filling Factors

We here consider the simplest case of uniform steady states at even-integer global filling factors and obtain the formula of the critical electric field \tilde{E}_{c1} as well as the value of \tilde{T}_{c1} . In the present case we have $\mu_r = 0$ everywhere in the sample. We consider only the two Landau levels near μ , neglect the spin splitting, and then use the simplified formula of $\tilde{\sigma}_{xx} = C_h f_1 (1 - f_1)$. We will give only a summary of the results, which we have obtained in our previous paper.²⁰⁾ The equation of the energy conservation in this case gives the balance of the energy gain \tilde{P}_G and the energy loss \tilde{P}_L :

$$(\tilde{P}_G =) \tilde{\sigma}_{xx} \tilde{E}^2 = C_p \tilde{T}_e (\tilde{P}_L) . \quad (5.1)$$

\tilde{P}_G and \tilde{P}_L are plotted as a function of \tilde{T}_e in Fig. 1(a). Points of intersection of the two curves, $\tilde{P}_G(\tilde{T}_e)$ and $\tilde{P}_L(\tilde{T}_e)$, give \tilde{T}_e in stationary states. The number of points of intersection increases from one to three as increasing \tilde{E} through a critical value. This critical value is \tilde{E}_{c1} , and is given by

$$\tilde{E}_{c1} = (C_p \gamma / C_h)^{1/2} \quad (\gamma = 2.2334) . \quad (5.2)$$

A bistability appears above \tilde{E}_{c1} (one of three in the middle corresponds to an unstable state). The present bistability originates from the existence of the activation energy $\hbar \omega_c / 2$ of \tilde{P}_G . \tilde{T}_e and $\tilde{\sigma}_{xx}$ in stationary states as a function of E/E_{c1} are plotted in Fig. 1(b). The value of

\tilde{T}_e at \tilde{E}_{c1} in the high \tilde{T}_e branch is \tilde{T}_{c1} , and is given by

$$\tilde{T}_{c1} = 0.324 . \quad (5.3)$$

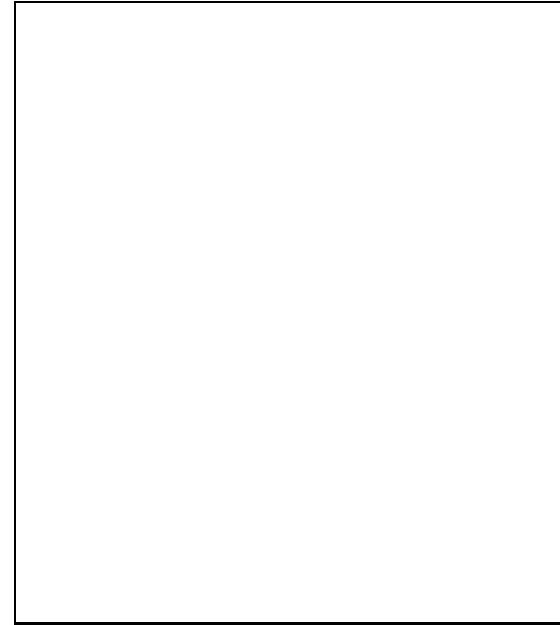


Fig. 1. (a) The energy gain $\tilde{P}_G = \tilde{\sigma}_{xx} \tilde{E}^2$ at the critical electric field E_{c1} and the loss $\tilde{P}_L = C_p \tilde{T}_e$ (both divided by C_p) as a function of the electron temperature \tilde{T}_e . Points of intersection give \tilde{T}_e in stationary states. (b) \tilde{T}_e and $\tilde{\sigma}_{xx}/C_h$ in stationary states as a function of E/E_{c1} .

5.2 Estimation of Coefficients

Rough estimates of the coefficients in the HDEQs at $B = 5T$ have been given in our previous papers^{20, 19)} and are summarized here:

$$C_p = 1.0 \times 10^{-5} , \quad C_h = 2.5 , \quad \tilde{\kappa}_{s0} = 0.016 . \quad (5.4)$$

The ratio of the two components of the thermal conductivities, $\tilde{\kappa}_h$ and $\tilde{\kappa}_s$, is then estimated to be

$$\tilde{\kappa}_h / \tilde{\kappa}_s = \frac{1}{4} C_h f_1 (1 - f_1) \tilde{T}_e^{-1} / \tilde{\kappa}_{s0} \approx 17 , \quad (5.5)$$

when $\mu_r = 0$ and $T_e = T_{c1}$. The thermal conductivity due to hopping processes is much larger than that due to electron-electron scatterings without hoppings. In the following we neglect $\tilde{\kappa}_s$.

The above estimates of C_p and C_h give theoretical estimates of \tilde{E}_{c1} and $\tilde{\sigma}_{xx}^s = \frac{1}{4} C_h$, the saturation value of $\tilde{\sigma}_{xx}$ at large T_e (only the two Landau levels near μ are considered). The theoretical estimates and the observed values^{3, 18)} are as follows.

$$\text{Theory}(B = 5.0T) : \tilde{E}_{c1} = 3 \times 10^{-3} , \quad \tilde{\sigma}_{xx}^s = 0.6 , \quad (5.6)$$

$$\text{Ref. 3}(B = 4.7T) : \tilde{E}_{c1} = 9 \times 10^{-3} , \quad \tilde{\sigma}_{xx}^s = 0.02 , \quad (5.7)$$

$$\text{Ref. 18}(B = 3.8T) : \tilde{E}_{c1} = 8 \times 10^{-3} , \quad \tilde{\sigma}_{xx}^s = 0.02 . \quad (5.8)$$

The discrepancies between the theory and the experiments will be within the limitation of accuracy in the

theory, since a large ambiguity in the estimation of ℓ_{vh}/ℓ produces large errors in the estimation of C_p and C_h . It may be convenient to consider C_p and C_h as adjustable parameters when spatio-temporal evolutions are compared between the theory and the experiment.

5.3 Scaled HDEQs

The energy gain and the energy loss in the HDEQs are very small in magnitude in the units we have used: $\tilde{P}_G \sim \tilde{\sigma}_{xx} \tilde{E}_{c1}^2 \sim 10^{-5}$ and $\tilde{P}_L \sim C_p \tilde{T}_{c1} \sim 10^{-5}$ according to the estimation in the previous subsection. Therefore we switch the units to the ones in which the length and the time are scaled by ℓ_{c1} and τ_{c1} , respectively, defined in §2.1.2. Scaled variables are defined as $\check{\mathbf{r}} = \mathbf{r}/\ell_{c1}$, $\check{\nabla} = \ell_{c1} \nabla$, $\check{t} = t/\tau_{c1}$, and $\check{\mathbf{E}} = e\ell_{c1} \mathbf{E}/\hbar\omega_c$. Ratios between the present scale and the previous scale are $\ell_{c1}/\ell = (\tilde{E}_{c1})^{-1}$ and $\tau_{c1}\omega_c = (\tilde{E}_{c1})^{-2}$. Since $\check{\mathbf{E}} = \tilde{\mathbf{E}}/\tilde{E}_{c1}$, the energy gain and loss become of the order of unity in the present units.

When $\mu_r = 0$ and only the two Landau levels near μ are considered for hopping processes, the HDEQs in the new units become

$$\frac{1}{\pi} \frac{\partial \tilde{\mu}_r}{\partial \tilde{t}} = -\check{\nabla} \cdot \check{\mathbf{j}}_n , \quad (5.9)$$

$$\tilde{C}_e \frac{\partial \tilde{T}_e}{\partial \tilde{t}} = -\check{\nabla} \cdot \check{\mathbf{j}}_U + \tilde{\sigma}_{xx} \check{E}^2 - C_h \gamma^{-1} \tilde{T}_e , \quad (5.10)$$

with

$$\check{\mathbf{j}}_n = (\ell_{c1}/\omega_c) \mathbf{j}_n = -\tilde{\sigma}_{xx} \check{\mathbf{E}} - \tilde{\sigma}_{xy} (\check{\mathbf{E}} \times \mathbf{e}_z) , \quad (5.11)$$

$$\check{\mathbf{j}}_U = (\ell_{c1}/\hbar\omega_c^2) \mathbf{j}_U = -\tilde{\kappa}_h \check{\nabla} \tilde{T}_e + \tilde{u}_{xy} (\check{\mathbf{E}} \times \mathbf{e}_z) . \quad (5.12)$$

The scaled HDEQs show that the typical length and time scales of variations of T_e and μ_r are ℓ_{c1} and τ_{c1} , respectively.

§6. Current Distribution at Even-Integer Filling Factors

In this section we illustrate how to apply the HDEQs to the calculation of spatial variations of T_e . We consider, as an example, the current distribution in a steady state in the higher- T_e branch in the middle of a long sample, in which spatial variations occur only perpendicular to the current. We restrict our calculation to the simple case where the global filling factor is $2N$ ($N = 1, 2, \dots$) and T_e is low enough that only the two Landau levels near μ contribute to \mathbf{j}_n^h , \mathbf{j}_U^h , and \mathbf{j}_U^d .

6.1 Electrostatics and Estimation of μ_r

In this subsection we derive the equation relating \mathbf{E} and μ_r . We show here that $\tilde{\mu}_r \sim (\tilde{E}_{c1})^2 \sim 10^{-5}$.

We define the two-dimensional local charge density by $\sigma_{\text{loc}}(\mathbf{r}) = -e(n_{\text{loc}} - n_0)$ with n_{loc} the local electron density and n_0 the uniform electron density corresponding to the global filling factor $2N$. The local charge density is divided into two components: $\sigma_{\text{loc}} = \sigma + \sigma_{\text{sr}}$ with σ long-ranged variations due to the Hall polarization and σ_{sr} short-ranged fluctuations induced by density fluctuations of ionized donors. The local electric field \mathbf{E}_{loc} is also divided into two components: $\mathbf{E}_{\text{loc}} = \mathbf{E} + \mathbf{E}_{\text{sr}}$ with \mathbf{E} the macroscopic electric field consisting of the Hall

field induced by σ and the uniform field along x and \mathbf{E}_{sr} the short-ranged fluctuations due to ionized donors and σ_{sr} . The long-ranged variations given by σ and \mathbf{E} are assumed to occur only along y ($0 < y < W$).

The dimensionless charge density $\tilde{\sigma} = \sigma \ell^2/e$ is given by

$$\tilde{\sigma} = -\frac{N}{\pi} \frac{\partial \tilde{E}_y}{\partial \tilde{y}} - \frac{1}{\pi} \tilde{\mu}_r . \quad (6.1)$$

The first term of $\tilde{\sigma}$ was given by MacDonald *et al.*²⁴⁾ It is the charge density in the presence of spatial modulations of the electric field when the lower N Landau levels are fully occupied and the other higher Landau levels are empty, and it comes from the polarization of wave functions of the occupied states. The second term is the charge density due to the occupation of electrons or holes, which is described by μ_r when averaged in a macroscopic scale.

The charge density $\tilde{\sigma}$ is also given by the Gauss law relating $\tilde{\sigma}$ and \tilde{E}_y :

$$\tilde{E}_y(\tilde{y}) = \frac{e^2/\epsilon\ell}{\hbar\omega_c} \int_0^{W/\ell} d\tilde{y}' \frac{2\tilde{\sigma}(\tilde{y}') \text{sgn}(\tilde{y} - \tilde{y}')}{[(\tilde{y} - \tilde{y}')^2 + \tilde{z}_c^2]^{1/2}} , \quad (6.2)$$

where ϵ is the dielectric constant and $\ell\tilde{z}_c$ is the cutoff length corresponding to the finite thickness of the electron density perpendicular to the layer. Here $\text{sgn}(y) = 1$ when $y > 0$ and $\text{sgn}(y) = -1$ when $y < 0$.

Now we estimate the order of magnitude of $\tilde{\mu}_r$. From eq.(6.2), we have

$$\begin{aligned} \frac{\partial \tilde{E}_y}{\partial \tilde{y}} &= \frac{e^2/\epsilon\ell}{\hbar\omega_c} \left[\frac{4\tilde{\sigma}(\tilde{y})}{\tilde{z}_c} - \int_0^{W/\ell} d\tilde{y}' \frac{2\tilde{\sigma}(\tilde{y}') |\tilde{y} - \tilde{y}'|}{[(\tilde{y} - \tilde{y}')^2 + \tilde{z}_c^2]^{3/2}} \right] \\ &\sim \frac{e^2/\epsilon\ell}{\hbar\omega_c} \frac{\tilde{\sigma}(\tilde{y})}{\tilde{z}_c} , \end{aligned} \quad (6.3)$$

where we have assumed that $\tilde{\sigma}(\tilde{y})$ does not change significantly in the scale of \tilde{z}_c . Then we have $\tilde{\sigma}(\tilde{y}) \sim \partial \tilde{E}_y / \partial \tilde{y}$ since $e^2/\epsilon\ell \sim \hbar\omega_c$ and $\tilde{z}_c \sim 1$. Combining this with eq.(6.1), we obtain $\tilde{\mu}_r \sim \partial \tilde{E}_y / \partial \tilde{y} \sim \tilde{E}_{c1} \ell / \ell_{c1} = (\tilde{E}_{c1})^2 \sim 10^{-5}$.

6.2 Macroscopic Variables and Equations

It was shown in the previous subsection that $\tilde{\mu}_r \sim (\tilde{E}_{c1})^2 \sim 10^{-5}$ when the system is uniform along x and the global filling factor is an even integer. Since $\tilde{\mu}_r \ll 1$ ($\mu_r \ll \hbar\omega_c$) and $\tilde{\mu}_r \ll \tilde{T}_e$ ($\mu_r \ll k_B T_e$), we neglect $\tilde{\mu}_r$ in the HDEQs and use eqs.(5.9), (5.10), (5.11), and (5.12). Macroscopic variables $T_e(y)$ and $E_y(y)$ are obtained by solving the two HDEQs, and $\mu_r(y)$, on the other hand, is determined by eqs.(6.1) and (6.2) in the previous subsection, using the obtained $E_y(y)$. In the present case, E_x does not depend on y , either, since $\partial E_x / \partial y = \partial E_y / \partial x = 0$. The constant value of E_x is determined by the total current through the system.

6.3 Equations for $T_e(y)$ and $E_y(y)$

The equation of the charge conservation in the present case is $\partial j_{ny} / \partial y = 0$, and the boundary condition is $j_{ny} = 0$ at sample edges ($y = 0, W$). Then we obtain

$$\check{j}_{nx} = -(\tilde{\rho}_{xx})^{-1} \check{E}_x , \quad (6.4)$$

$$\check{j}_{ny} = -\tilde{\sigma}_{yy}\check{E}_y - \tilde{\sigma}_{yx}\check{E}_x = 0 , \quad (6.5)$$

for $0 < y < W$, with $\tilde{\rho}_{xx} = \tilde{\sigma}_{xx}/(\tilde{\sigma}_{xx}^2 + \tilde{\sigma}_{xy}^2)$. The equation of the energy conservation in the present case is

$$\frac{\partial}{\partial \check{y}} \tilde{\kappa}_h(\tilde{T}_e) \frac{\partial}{\partial \check{y}} \tilde{T}_e = -\eta(\tilde{T}_e) \frac{\partial}{\partial \check{y}} \tilde{T}_e + F(\tilde{T}_e) , \quad (6.6)$$

$$\eta(\tilde{T}_e) = \frac{1}{2\pi} \check{E}_x f_1 (1 - f_1) \tilde{T}_e^{-2} , \quad (6.7)$$

$$F(\tilde{T}_e) = -\tilde{\rho}_{xx}^{-1} \check{E}_x^2 + C_h \gamma^{-1} \tilde{T}_e . \quad (6.8)$$

Here we have used eq.(6.5) to eliminate \check{E}_y . The boundary condition is $j_{Uy} = 0$ at $y = 0, W$, and is given by

$$\tilde{\kappa}_h \frac{\partial}{\partial \check{y}} \tilde{T}_e = -\frac{1}{\pi} f_1 \check{E}_x . \quad (6.9)$$

6.4 Current Density Modulation

The above boundary condition eq.(6.9) shows that T_e depends on y due to a nonzero E_x . This modulation of T_e gives a modulation of $j_x = (-e)j_{nx}$, since $\tilde{\rho}_{xx}$ in eq.(6.4) depends on T_e (when $\tilde{\rho}_{xx}$ is a constant, on the other hand, j_x becomes uniform as pointed out by Thouless²⁵⁾). The present current density modulation originates from the drift component of the energy flux perpendicular to \mathbf{E} (the second term in eq.(5.12)). This energy flux has nonzero y component when $E_x \neq 0$ and produces the modulation of T_e in eq.(6.9), and consequently the modulation of j_x .

Since the equation for \tilde{T}_e is nonlinear, numerical calculations are necessary to obtain the solution, which will be performed in the future work. Here we instead discuss the qualitative profile of T_e and j_x by regarding eq.(6.6) as an equation of motion of a particle with the “position” \tilde{T}_e as a function of the “time” \check{y} . In this mechanical analogue, $\tilde{\kappa}_h$ is the “mass”, η is the “friction coefficient”, and F is the “force”. The corresponding “potential” around the higher- T_e stationary point is plotted schematically in Fig. 2(a). The boundary condition eq.(6.9) gives the “velocity” of the particle at the “time” $\check{y} = 0, W/\ell_{c1}$.

When W is large, the particle spends the most of its “time” at the top of the “potential”. This means that $T_e(y)$ and $j_x(y)$ are approximately uniform in the middle of the sample, as shown schematically in Fig. 2(b). The length scale for the relaxation of $T_e(y)$ and $j_x(y)$ from their deviations at the edges is of the order of $\ell_{c1} \sim 2\mu\text{m}$.

In the QHE regime with negligible ρ_{xx} , MacDonald *et al.*²⁴⁾ have shown that $j_x(y)$ is enhanced near the both edges, because of the edge electrostatics in the 2DES. When ρ_{xx} is large, on the other hand, we find in Fig. 2(b) that $j_x(y)$ in one edge is enhanced, while $j_x(y)$ in the other edge is suppressed. This comes again from the boundary condition eq.(6.9) stating that the derivative of $T_e(y)$ has the same sign in the two edges.

§7. Conclusions and Discussions

We have derived hydrodynamic equations (HDEQs) for quantum Hall systems in the regime of large energy dissipation. We have shown that the strong magnetic field in the present systems produces a component of the energy flux which is perpendicular to the electric field. The presence of this energy flux is the most important feature of our HDEQs for the quantum Hall systems in

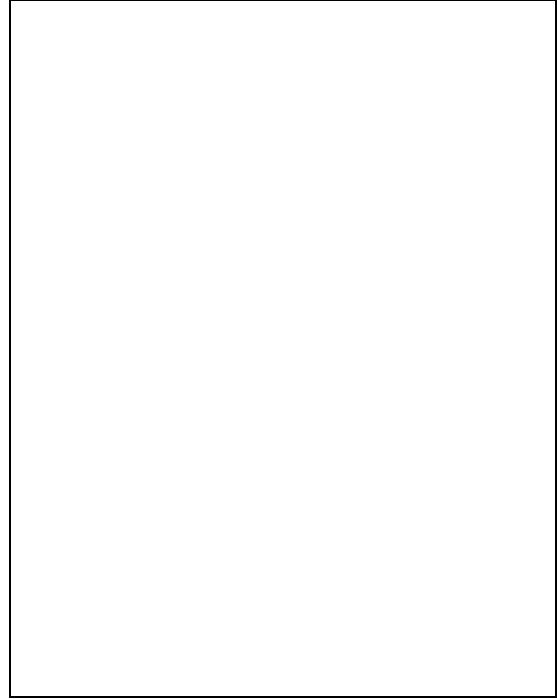


Fig. 2. (a) “Potential” in a mechanical analogue as a function of the “position” \tilde{T}_e in the vicinity of the higher- T_e stationary point. (b) Schematic spatial variations of the electron temperature T_e and the current density j_x in the sample (shaded region) when $j_x > 0$.

the high-electron-temperature regime. This energy flux is absent in the equation proposed by the previous authors.¹⁷⁾ The HDEQs with this unique energy flux have predicted a new type of current density modulation.

Another example of the application of the HDEQs is to investigate spatial evolutions of T_e in the current direction (along x) when a spatial transition occurs between the small ρ_{xx} -state and the large ρ_{xx} -state along x . The observed spatial evolutions of ρ_{xx} in such transitions have exhibited a divergence of the relaxation length as the current density approaches a critical value.^{10, 11, 12)} In our previous paper,¹⁹⁾ we have applied the previous one-dimensional HDEQ to this problem and shown that the calculated relaxation length also exhibits a divergence qualitatively similar to the observed one.

A hydrodynamic model²⁶⁾ has been proposed to explain the observed longitudinal voltage steps.^{27, 28)} It is argued in this model that electron-hole pairs created at inter-Landau-level scatterings near a charged impurity have common features with vortices created when a classical fluid flows past an obstacle. The hydrodynamic equations in the present paper, on the other hand, describe macroscopic spatio-temporal variations, and their typical length scale is much larger than the size of a single electron-hole pair.

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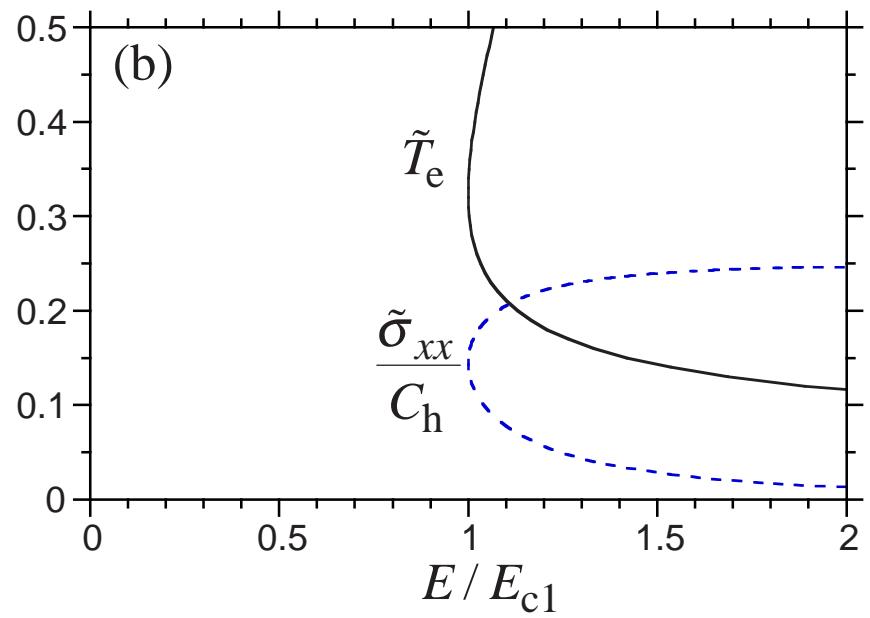
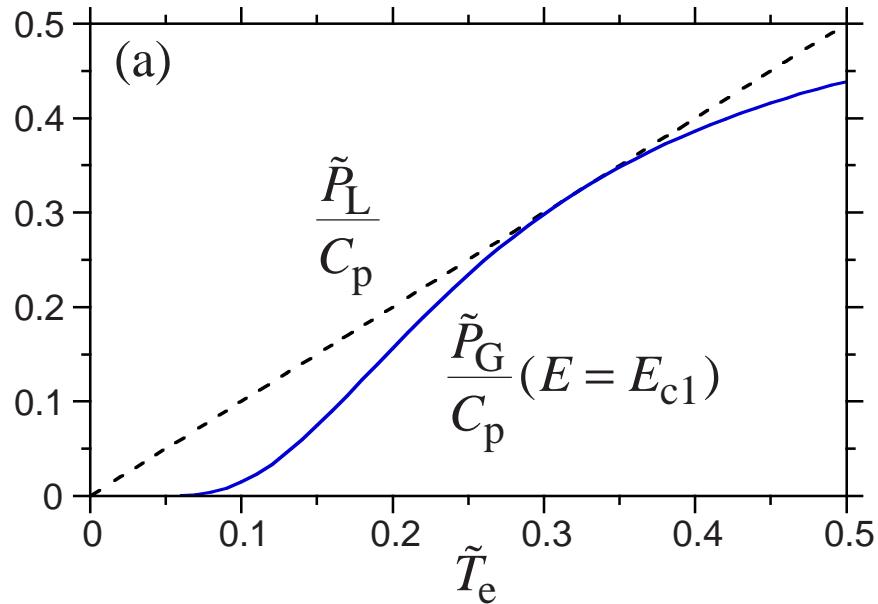


Fig.1

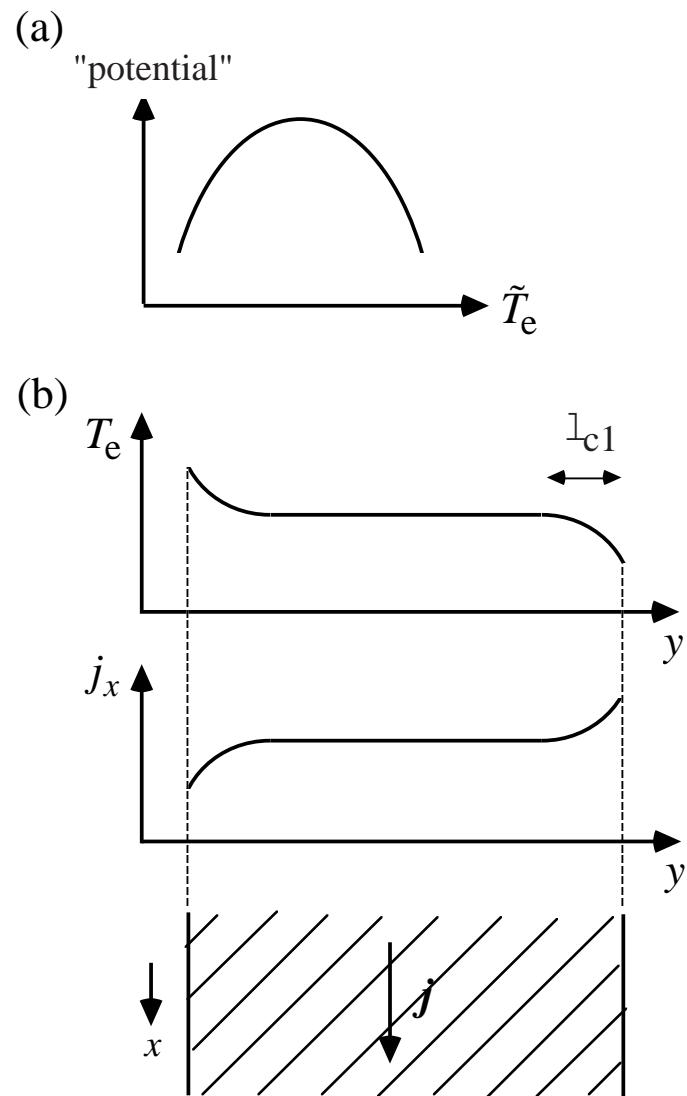


Fig. 2